Equation (25) relates t to T_0 through $t = t(\gamma)$ and Eq. (23). The solutions for $\dot{S}$ and δ are

$$\dot{S} = \dot{S}_1 + \dot{S}'_2(T_f - T_\infty)\gamma \tag{28}$$

$$\delta = \frac{2k(T_f - T_\infty)\gamma}{h(T_f - T_\infty)(1 - \gamma) - \rho L_e \dot{S}} \tag{29}$$

For the case of $\dot{S}'_2 = 0$, the original governing equations become linear and the exact solution is known.[2] It is found that the present solution agrees very well with the exact solution.

4. Erosion with Ablation

Once the surface has reached an ablation temperature, Eqs. (8, 14, 15, 17, and 18) are applicable. Since the surface temperature is now known, Q_0 and $\dot{S}_e$ are functions of time only. Consider again the case of constant external conditions, so that Q_0 and $\dot{S}_e$ are constant. Then Eq. (8) can be integrated to yield

$$t - t_a = \int_{\delta a}^{\delta} \frac{\delta}{3} d\delta \Bigg/ \left[2\alpha_0\left(1 + \frac{\theta_0}{\rho_0 L_a}\right) - \left(\dot{S}_e + \frac{Q_0}{\rho_0 L_a}\right)\delta \right] \tag{30}$$

where t_a is the time when ablation starts and δ_a is the value of δ at $t = t_a$. For simplicity, consider constant properties and let $Q_0 = F - \rho L_e \dot{S}_e$. Equation (30) yields

$$t - t_a = \frac{N(\delta_a - \delta) - \ln[(1 - N\delta)/(1 - N\delta_a)]}{6\alpha[1 + c(T_a - T_\infty)/L_a]N^2} \tag{31}$$

and Eqs. (14) and (18) yield

$$S = S_a + \frac{\delta_a - \delta}{3} - \frac{\ln[(1 - N\delta)/(1 - N\delta_a)]}{3N[1 + c(T_a - T_\infty)/L_a]} \tag{32}$$

where S_a is the value of S at $t = t_a$ and

$$N = \frac{\dot{S}_e(1 - L_e/L_a) + F/\rho L_a}{2\alpha[1 + c(T_a - T_\infty)/L_a]} \tag{33}$$

It may be noted here that as $t \to \infty$, δ and $\dot{S}$ approach the steady-state values of $\delta \to 1/N$ and $\dot{S} \to 2\alpha N$.

It can easily be shown that when $\dot{S}_e = 0$, Eqs. (31) and (32) reduce to the solution for phase change alone as obtained by Goodman.[1] Examination of the results shows that for $\dot{S}_e \neq 0$, the present solution can be equated to Goodman's solution by letting F of his solution be $F + \rho(L_a - L_e)\dot{S}_e$ for the present solution, and by adjusting his integration constant to account for the difference in δ_a. Thus, in the special case when the heat of erosion is equal to the heat of ablation, the two solutions have identical form. It has been stated by Goodman that his results are impossible to distinguish, on a plotted scale, from the exact numerical solution of Landau.[3] One may infer from this statement that the present results are also quite accurate.

5. Discussion

The formulation presented in Sec. 2 allows one to determine the transient temperature distribution, erosion rate, and ablation rate under general erosive environments and arbitrary heating conditions. The method of solution consists of solving a first-order ordinary nonlinear differential equation along with several nonlinear algebraic equations. The numerical computation can be carried out quite simply. The closed-form solutions presented in Secs. 3 and 4 illustrate the general features of heat conduction with an eroding surface.

It is noted that while condensed phase changes usually occur at a unique temperature, erosion may occur over a wide range of surface temperatures. Also, the rate of erosion is in general a prescribed function of surface temperature and time, but the rate of phase change is determined by a heat balance at the surface. Furthermore, erosion and phase change may occur simultaneously and can be treated as shown.

For erosion problems involving pulselike environments, finite slabs, etc., the special techniques illustrated by Goodman[1] may be employed. For erosion rates that depend on surface heat flux and "char thickness" as well as surface temperature and time, the integral approach is an ideal one since both heat flux and char thickness can in turn be related to the surface temperature and heat penetration distance. Finally, it may be noted that if a different representation, such as a higher order polynomial or an exponential profile, is used for the temperature distribution, the governing differential equation will retain the same form with modified coefficients.

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Stability of Linear Regulators Optimal for Time-Multiplied Performance Indices

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I. Introduction

It is well-known that the optimal control law for the time-invariant linear feedback system,

$$\dot{x} = Fx + Gu, \quad x(0) = x_0 \tag{1}$$

which minimizes the quadratic performance index

$$J = \int_0^\infty (x^T Q x + u^T u) dt \tag{2}$$

is given by

$$u^* = -K^T x = -(PG)^T x \tag{3}$$

and P is defined as the solution of

$$-\dot{P} = PF + F^T P - PGG^T P + Q \tag{4}$$

with the initial condition

$$\lim_{T \to \infty} P(T) = 0 \tag{5}$$

Anderson and Moore[1] have shown under assumptions of complete controllability and complete observability that a large amount of nonlinearity may be tolerated in the optimal control law without causing the system to become unstable. In Ref. 2, this result is extended to systems optimal with respect

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to performance index of the type

$$J = \int_0^\infty [m(x) + u^T u]dt \tag{6}$$

where $m(x)$ is assumed non-negative for all x.

This paper extends the aforementioned results to the case where the system (1) is optimum for the performance index given by either

$$J_k = \int_0^\infty (t^k x^T Q_0 x + u^T u)dt \tag{7}$$

where Q_0 is a positive definite matrix and k is any non-negative integer or

$$J_k = \int_0^\infty [t^k m_0(x) + u^T u]dt \tag{8}$$

where $m_0(x)$ is a positive definite radially unbounded function of x and k is as before.

A useful lemma

The following lemma the proof of which follows closely that of a related lemma by MacFarlane[3] is used in the sequel.

Lemma: Consider an asymptotically stable time-invariant system

$$\dot{x} = F(x), \quad x(0) = x_0 \tag{9}$$

where $F(x)$ is a nonlinear function of x, and a time multiplied performance index

$$I_k = \int_0^\infty t^k m_0(x)dt \tag{10}$$

where $m_0(x)$ is a positive definite, radially unbounded function of x. Then Eq. (10) is equivalent to

$$I_k = \int_0^\infty k!m_k(x)dt \tag{11}$$

and is given by

$$I_k = -k!m_{k+1}(x)|_{t=0}^{\infty} = k!m_{k+1}(x_0) \tag{12}$$

with

$$(d/dt)m_{r+1}(x) = -m_r(x), \quad r = 0,1,\ldots,k \tag{13}$$

where the indicated derivative is taken along the trajectories of the system (13). Proof of the lemma is given in the appendix.

Corollary: For the special case where $m_0(x) = x^T Q_0 x$ with Q_0 positive definite and the system is linear time-invariant satisfying

$$\dot{x} = Fx, \quad x(0) = x_0 \tag{14}$$

Equations (11–13) reduce to

$$I_k = \int_0^\infty k!x^T Q_k x dt \tag{15}$$

and is given by

$$I_k = k!x_0^T Q_{k+1} x_0 \tag{16}$$

where Q_r satisfies the following sequence of linear matrix equations:

$$F^T Q_{r+1} + Q_{r+1} F + Q_r = 0, \quad r = 0,1,\ldots,k \tag{17}$$

II. Main Results

a Time multiplied quadratic performance index

The optimal control law for the system (1) minimizing the performance index (7) is derived in Ref. 4 using the aforementioned corollary and is given by

$$u^* = -G^T Q_{k+1} x = -K^T x \tag{18}$$

where Q_{k+1} is the positive definite solution of the matrix Ricatti equation;

$$-\dot{Q}_{k+1} = F^T Q_{k+1} + Q_{k+1} F - Q_{k+1} G G^T Q_{k+1} + k!Q_k \tag{19}$$

and Q_r is the positive definite solution of the algebraic matrix linear equation

$$(F - GK^T)^T Q_{r+1} + Q_{r+1}(F - GK^T) = 0, \quad r = 0,1,\ldots,k-1 \tag{20}$$

Theorem 1: The system (1) optimized with respect to the performance index (7) remains stable even after the optimal control law $u^* = -K^T x = -G^T Q_{k+1} x$ is changed to either

$$\text{(i) } u^* = -\tfrac{1}{2}K^T x - \psi \tag{21}$$

where ψ is a nonlinear vector function of $K^T x$, satisfying

$$\sigma^T \psi(\sigma) > 0 \text{ with } \sigma = K^T x, \quad \sigma \neq 0 \tag{21a}$$

or

$$\text{(ii) } \quad u^* = -k(t)K^T x \tag{22}$$

with, for all t,

$$\tfrac{1}{2} + \epsilon \leq k(t) \leq \bar{k} < \infty \tag{22a}$$

with $\bar{k}$ arbitrary.

Proof: (i) The stability of the closed loop system for the control given by Eq. (21) can be demonstrated by taking as a Lyapunov function

$$V(x,t) = x^T(t)Q_{k+1}(t)x(t) \tag{23}$$

$$\dot{V} = (x^T F^T - \tfrac{1}{2}x^T K G^T - \psi^T G^T)Q_{k+1}x + x^T \dot{Q}_{k+1} x + x^T Q_{k+1}(Fx - \tfrac{1}{2}GK^T x - G\psi)$$

Substituting for $\dot{Q}_{k+1}$ from Eq. (19) leads to

$$\dot{V} = -k!x^T Q_k x - 2\psi^T(K^T x)$$

Equation (21a) guarantees that $\dot{V}$ is nonpositive and thus stability of the closed loop system follows.

(ii) The control in Eq. (22) can be written as

$$u^* = -\tfrac{1}{2}K^T x - k_1(t)K^T x \tag{24}$$

where $0 < k_1(t) \leq \bar{k}_1 < \infty$. For this control, the derivative is

$$\dot{V} = [x^T F^T - \tfrac{1}{2}x^T K G^T - k_1(t)x^T K G^T]Q_{k+1}x + x^T \dot{Q}_{k+1} x + x^T Q_{k+1}[Fx - \tfrac{1}{2}GK^T x - k_1(t)GK^T x] = -k!x^T Q_k x - 2k_1(t)x^T Q_{k+1} G G^T Q_{k+1} x$$

Equation (24) guarantees that $\dot{V}$ is nonpositive, and thus stability of closed loop system follows.

b Nonquadratic index

The optimal control law for the system (1) minimizing the performance index (8) is derived based on the assumption that $m_0(x)$ is such that an optimal control law exists which makes the closed loop system asymptotically stable. Then, using the lemma, the performance index given in Eq. (8) becomes

$$J_k = \int_0^\infty [k!m_k(x) + u^2]dt \tag{25}$$

with

$$(d/dt)m_{r+1}(x) = -m_r(x), \quad r = 0,1,\ldots,k \tag{26}$$

along the trajectories $\dot{x} = Fx + Gu$, $x(0) = x_0$.

Let $J_k^* = n(x)$ be the optimal performance index. Then the optimal control law is

$$u^* = -\tfrac{1}{2}G^T \nabla J_k^* = -\tfrac{1}{2}G^T \nabla n(x) \tag{27}$$

and Hamilton-Jacobi equation is

$$x^T F^T \nabla n(x) - \tfrac{1}{4}\nabla^T n(x) G G^T \nabla n(x) + k!m_k(x) = 0 \tag{28}$$

Theorem 2: The system (1) optimized with respect to the performance index (8) remains stable even after the optimal control law $u^* = -\frac{1}{2}G^T\nabla n(x)$ is changed to either

$$\text{(i)} \quad u^* = -\tfrac{1}{4}G^T\nabla n(x) - \psi \tag{29}$$

where ψ is a nonlinear vector function of $\frac{1}{2}G^T\nabla n(x)$, satisfying $\sigma^T\psi(\sigma) > 0$ with

$$\sigma = \tfrac{1}{2}G^T\nabla n(x), \quad \sigma \neq 0 \tag{29a}$$

or

$$\text{(ii)} \quad u = -\tfrac{1}{2}k(t)G^T\nabla n(x) \tag{30}$$

with, for all t,

$$\tfrac{1}{2} + \epsilon \leq k(t) \leq \bar{k} < \infty \tag{31}$$

with $\bar{k}$ arbitrary.

Proof: (i) The closed loop system after the introduction of nonlinearity in the optimal control is

$$\dot{x} = Fx - \tfrac{1}{4}GG^T\nabla n(x) - G\psi \tag{32}$$

Using as the Lyapunov function $V = n(x)$

$$\dot{V} = \nabla^T n(x)[Fx - \tfrac{1}{4}GG^T\nabla n(x) - G\psi] =$$
$$x^TF^T\nabla n(x) - \tfrac{1}{4}\nabla^T n(x)GG^T\nabla n(x) - G\psi =$$
$$-k!m_k(x) - G^T\nabla n(x)\psi$$

the last line following by use of the Hamilton-Jacobi Eq. (28). Evidently $\dot{V}$ is always nonpositive establishing stability.

(ii) Proof in the case when time variation is introduced into the optimal control law follows easily. It can be seen that Theorem 2 is valid even when the system is nonlinear, i.e., $\dot{x} = F(x,u)$ where $F(x,u)$ is a nonlinear function of x and u, provided an optimal control exists which makes the closed loop system asymptotically stable.

Appendix

Proof of the lemma

Consider $(d/dt)[t^l m_{j+1}(x)]$ for $l,j = 0,1,\ldots,k$ along the trajectories of Eq. (9).

$$(d/dt)[t^l m_{j+1}(x)] = lt^{l-1}m_{j+1}(x) + t^l(d/dt)m_{j+1}(x) =$$
$$lt^{l-1}m_{j+1}(x) - t^l m_j(x) \tag{A1}$$

assuming

$$(d/dt)[m_{j+1}(x)] = -m_j(x), \quad j = 0,1,\ldots,k \tag{A2}$$

Since the system (9) is asymptotically stable, (A2) gives a positive definite $m_{j+1}(x)$ for positive definite $m_j(x)$. From Eq. (A1)

$$\int_0^\infty t^l m_j(x)dt = \int_0^\infty lt^{l-1}m_{j+1}(x)dt - t^l m_{j+1}(x)\Big|_{t=0}^{\infty} =$$
$$\int_0^\infty lt^{l-1}m_{j+1}(x)dt$$

since Eq. (9) is asymptotically stable. Therefore

$$\int_0^\infty t^k m_0(x)dt = \int_0^\infty kt^{k-1}m_1(x)dt$$
$$= \int_0^\infty k(k-1)t^{k-2}m_2(x)dt$$
$$\vdots$$
$$= \int_0^\infty k!m_k(x)dt$$
$$= -\int_0^\infty k!\frac{d}{dt}m_{k+1}(x)dt$$
$$= k!m_{k+1}(x_0) \tag{A3}$$

where

$$(d/dt)m_{j+1}(x) = -m_j(x), j = 0,1,\ldots,k \tag{A4}$$

along the trajectories of Eq. (9).

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Transient Behavior of Charring Ablators

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INTEREST in thermal protection devices for very high-speed re-entry has led to the study of charring ablators. The behavior of a phenolic nylon ablator subjected to transient heat inputs has recently been investigated experimentally.[1] These data indicate the average pyrolysis mass-loss rate to be greater under increased heating than decreased heating. It was then concluded that "there is an intrinsic difference between material behavior in increasing and in decreasing heating." This result provided the impetus for a theoretical calculation using a simplified model of the transient behavior of charring ablators.

The one-dimensional transient heat-transfer model employed is similar to that of Barriault and Yos[2] and Adarkar and Hartsook.[3] The decomposition and chemical interactions of the original ablator material (virgin) are characterized in the current model by a constant temperature decomposition, absorption of heat (heat of pyrolysis), yielding of a residue matrix (char), and evolving gaseous products (pyroly-

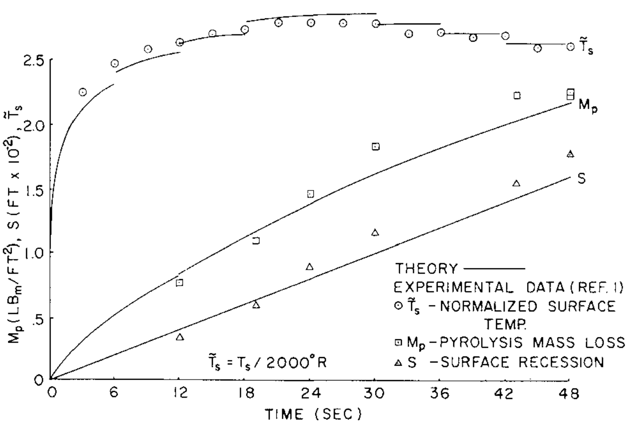

Fig. 1 Transient heating results for direct cycle.

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